On the Primitive Substitution Groups of Degree Sixteen.

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The object of this paper is to determine all the primitive groups of degree 16 and to study them in regard to solvability. It will be seen that the number of these groups is considerably larger than it has been supposed to be.* The paper is divided into two sections. In the first we determine the groups by very simple methods and study them with respect to the given property. In the second we prove that it is impossible to construct a primitive group of degree 16 that is not given in the first section.

§1.

Determination of the groups.

It will be proved that all these groups, with the exception of the two that include the alternating group of this degree, contain a self-conjugate subgroup of order 16. As a self-conjugate subgroup of a primitive group is transitive, this must be regular. Since the entire group must transform each of its substitutions except identity into a system that generates it, it must be the Abelian group that contains 15 subgroups of order 2. We shall denote it by H and suppose that its substitutions are

1		*ai . bj . ck . dl . em . fn . go . hp*	*I*,
ab . cd . ef . gh . ij . kl . mn . op	*B*,	*aj . bi . cl . dk . en . fm . gp . ho*	*J*,
ac . bd . eg . fh . ik . jl . mo . np	*C*,	*ak . bl . ci . dj . eo . fp . gm . hn*	*K*,
ad . bc . eh . fg . il . jk . mp . no	*D*,	*al . bk . cj . di . ep . fo . gn . hm*	*L*,
ae . bf . cg . dh . im . jn . ko . lp	*E*,	*am . bn . co . dp . ei . fj . gk . hl*	*M*,
af . be . ch . dg . in . jm . kp . lo	*F*,	*an . bm . cp . do . ej . fi . gl . hk*	*N*,
ag . bh . ce . df . io . jp . km . ln	*G*,	*ao . bp . cm . dn . ek . fl . gi . hj*	*O*,
ah . bg . cf . de . ip . jo . kn . lm	*H*,	*ap . bo . cn . dm . el . fk . gj . hi*	*P*.

* Only 12 of these groups that do not contain the alternating group are given in Jordan's enumeration, Comptes rendus, vol. 75, p. 1757. We shall prove the existence of 20 such groups.

The group of isomorphisms of H* is evidently doubly transitive, of degree 15 and order $15 \,.\, 14 \,.\, 12 \,.\, 8 = 8\,! \div 2$. It is known that there is only one group that satisfies these conditions, and that it is simply isomorphic to the alternating group of degree 8 (K). To every subgroup of K corresponds a transitive group of degree 16 that contains H as self-conjugate subgroup. Its subgroup that includes all its substitutions which do not involve a given element is simply isomorphic to the corresponding subgroup of K. We shall confine our attention to the primitive groups.

The subgroup of the group of isomorphism that corresponds to a primitive group must transform each of the substitutions of H, excepting identity, into substitutions that generate H. It must therefore be of degree 15, and its order cannot be less than 5. To a subgroup of order 5 in K corresponds a group of order 80 that contains H as self-conjugate subgroup. This group (G_1) must be primitive, for it contains no self-conjugate subgroup besides H and 1, and there is no transitive group of degree 8 and order 80. It is solvable since its factors of composition are 2, 2, 2, 2, 5. It may be generated by H and

$$boejc \,.\, dpknl \,.\, fhgim \,.†$$

From the preceding paragraph it follows that to every subgroup of K whose order is divisible by 5 there corresponds a primitive group of degree 16 that contains H as self-conjugate subgroup. If we include K, there are 16 such subgroups, viz. $(abcde)$ cyc.,‡ $(abcde)_{10}$, $(abcde)$ cyc. (fgh) cyc., $\{(abcde)_{20}\,(fg)\}$ pos., $(abcde)_{10}\,(fgh)$ cyc., $(abcde)$ pos., $(abcdef)_{60}$, $\{(abcde)_{20}\,(fgh)$ all$\}$ pos., $\{(abcde)$ all $(fg)\}$ pos., $\{(abcdef)_{120}\,(gh)\}$ pos., $(abcde)$ pos. (fgh) cyc., $(abcdef)$ pos., $\{(abcde)$ all (fgh) all$\}$ pos., $\{(abcdef)$ all $(gh\}$ pos., $(abcdefg)$ pos., $(abcdefgh)$ pos. As all the subgroups of K that are similar to any one of these are conjugate, there can be no more than 16 primitive groups of degree 16 that contain H as self-conjugate subgroup and whose orders are divisible by 5. We proceed to prove that no two of these 16 primitive groups are simply isomorphic and to find their factors of composition and generating substitutions.

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* Hölder, Mathematische Annalen, vol. 43, p. 314.

† The substitutions that are to be added to H to generate the required groups permute the substitutions of H in exactly the same manner as their own elements; i. e. they have been selected in such a way they they are the same as the corresponding ones in capital letters.

‡ Cayley, Quarterly Journal of Mathematics, vol. 25, p. 71.

The group of order 160 (G_2) may be generated by G_1 and

$$bj.dl.eo.fh.gm.np.$$

Since it contains G_1 as maximal self-conjugate subgroup, its factors of composition may be found by adding 2 to those of G_1. Hence it is solvable. The group of order 240 (G_3) may be generated by G_1 and

$$bmn.cik.deh.flo.gpj.$$

As it contains G_1 as maximal self-conjugate subgroup, its factors of composition may be obtained by adding 3 to those of G_1. It evidently contains 16 conjugate subgroups of order 15. The groups of orders 320 and 480 (G_4, G_5) may be generated, respectively, by G_2 and

$$bejo.dglm.fnhp.ik, \quad bmn.cik.deh.flo.gpj.$$

Since G_2 is a maximal subgroup of each of these groups, their factors of composition may be obtained by adding 2, 3 respectively to those of G_2. Hence they are solvable. G_5 contains G_3 as self-conjugate subgroup.

Since K contains 3 subgroups of order 60, there are 3 primitive groups of degree 16 and order 960 that contain H as self-conjugate subgroup. Only one of these (G_6) is solvable. It may be generated by G_5 and

$$bejo.dglm.fnhp.ik.$$

It is solvable since its factors of composition may be obtained by adding 2 to those of G_5. We have now found the 6 solvable primitive groups of degree 16 whose orders are divisible by 5. The remaining 12 groups whose orders satisfy this condition are insolvable.

The remaining two groups of order 960 correspond to

$$(abcdef)_{60}, \quad (abcde) \text{ pos.}$$

in K. Hence each of them contains only two self-conjugate subgroups, viz. H and 1. Their factors of composition are 2, 2, 2, 2, 60. The former (G_7) may be generated by G_2 and

$$bpo.cnd.ekl.ghi.$$

The latter (G_8) may be generated by H and

$$boifc.dpgnh.emjlk, \quad bop.cge.dil.fmj.hkn.$$

Since the substitutions of order 3 in G_7 permute only 12 of the substitutions of H while those of G_8 permute 15, these two groups are not simply isomorphic.

The two primitive groups of order 1920 (G_9, G_{10}) correspond to

$$\{(abcdef)_{120}(gh)\} \text{ pos.,} \quad \{(abcde) \text{ all } (fg)\} \text{ pos.}$$

in K. Hence each of them contains 3 self-conjugate subgroups, viz. H, 1 and the one of order 960. Their factors of composition are obtained by adding 2 to those of the preceding two groups. They may be generated by adding

$$bdjl\,.\,ck\,.\,epon\,.\,fmhg, \quad bpch\,.\,di\,.\,ejmk\,.\,fgon$$

to G_7 and G_8 respectively. They cannot be simply isomorphic since their self-conjugate subgroups of order 960 do not have this property.

The group of order 2880 (G_{11}) may be generated by G_8 and

$$bmn\,.\,dil\,.\,ceg\,.\,hoj\,.\,fkp\,.$$

Its factors of composition may evidently be obtained by adding 3 to those of G_8. The two groups of order 5760 (G_{12}, G_{13}) correspond to

$$\{(abcde) \text{ all } (fgh) \text{ all}\} \text{ pos.,} \quad (abcdef) \text{ pos.}$$

Since the former contains G_{11} as self-conjugate subgroup, its factors of composition may be obtained by adding 2 to those of G_{11}. This is the last of the six primitive groups of degree 16 whose solution depends only upon that of the alternating group of degree 5. It may be generated by G_{11} and

$$bpch\,.\,di\,.\,ejmk\,.\,fgon\,.$$

The factors of composition of G_{13} are evidently 2, 2, 2, 2, 360. It may be generated by G_7 and

$$bop\,.\,cge\,.\,dil\,.\,fmj\,.\,hkn\,.$$

For

$$bnloj\,.\,cpekd\,.\,fhimg \times bj\,.\,dl\,.\,eo\,.\,fh\,.\,gm\,.\,np = bpo\,.\,cnd\,.\,ekl\,.\,ghi,$$
$$bpo\,.\,ceg\,.\,dli\,.\,fjm\,.\,hnk \times bpo\,.\,cnd\,.\,ekl\,.\,ghi = bop\,.\,cki\,.\,deh\,.\,fjm\,.\,gnl\,.$$

Hence the given generator corresponds to a substitution in the same elements in K as G_7. It clearly corresponds to a substitution of degree 3.

The group of order 11520 (G_{14}) contains G_{13} as self-conjugate subgroup. Its factors of composition may therefore be obtained by adding 2 to those of G_{13}. It is generated by G_{13} and

$$bdjl\,.\,ck\,.\,epon\,.\,fmhg\,.$$

G_{13} and G_{14} are the only two primitive groups of degree 16 whose solution depends only upon that of the alternating group of degree 6 but not upon any one of its subgroups.

The group (G_{15}) which corresponds to the alternating group of degree 7 in K may be generated by H and

$$bioplhjg \,.\, clnmekf, \quad bcd \,.\, emi \,.\, fol \,.\, gpj \,.\, hnk.$$

For the former of these generators is transformed into its square by

$$eln \,.\, fkm \,.\, gjp \,.\, hio.$$

Since the latter is commutative to this, it must correspond to a substitution of degree 3 in K whose elements are included in the substitution of degree 7 to which the former generator corresponds. The factors of composition of G_{15} are 2, 2, 2, 2, 2520.

The largest group that contains H as self-conjugate subgroup (G_{16}) may be generated by G_{15} and

$$bd \,.\, fh \,.\, jl \,.\, np.$$

For the alternating group of degree 8 contains only two types of substitutions of order 2. We have seen that the substitutions which correspond to those of the type $ab \,.\, cd$ permute 12 substitutions of H. The given generator must therefore correspond to a substitution of degree 8 in K. The factors of composition of G_{16} are 2, 2, 2, 2, 20160.

We have now considered all the possible primitive groups of degree 16 that contain H as a self-conjugate subgroup and whose order is divisible by 5. We have seen that 6 of these 16 groups are solvable, 6 others depend upon the solution of the alternating group of degree 6, while the remaining four depend upon the solution of the alternating groups of degrees 6, 7 and 8. There are 4 additional primitive groups that contain H as a self-conjugate subgroup. We proceed to consider these.

Since the groups

$(ae \,.\, bf \,.\, cg \,.\, dh)(abc)$ cyc. (efg) cyc.,* $\quad (ae \,.\, bf \,.\, cg \,.\, dh)\{(abc)$ all (efg) all$\}$ pos.,
$(afbe \,.\, cg \,.\, dh)\{(abc)$ all (efg) all$\}$ pos., $\quad (ae \,.\, bf \,.\, cg \,.\, dh)(abc)$ all (efg) all

are maximal subgroups that do not contain any self-conjugate subgroup besides

* Cayley, loc. cit.

identity of the four groups, in order,

$(ae.bf.cg.dh)(abcd)$ pos. $(efgh)$ pos., $(ae.bf.cg.dh)\{(abcd)$ all $(efgh)$ all$\}$ pos.,
$(afbe.cg.dh)\{(abcd)$ all $(efgh)$ all$\}$ pos., $(ae.bf.cg.dh)(abcd)$ all $(efgh)$ all,

each of the latter four groups is simply isomorphic to a primitive group of degree 16.* As each of these groups is evidently solvable, its factors of composition are the same as the prime factors of its order.

The first (G_{17}) may be generated by H and

$$bjofpm.ceg.dnihlk, \quad bpo.cnd.ekl.ghi.$$

Hence all its self-conjugate subgroups, except identity, contain H. From this it follows independently that it is primitive. G_{18} may be generated by G_{17} and

$$bmpjof.cki.dghlen.$$

It contains G_{17} as self-conjugate subgroup. It is of order 576 while G_{17} is of order 288. G_{19} is of the same order as G_{18}. It may be generated by H and

$$bpo.cnd.ekl.ghi, \quad bm.cide.fojp.gklh.$$

G_{20} contains G_{17}, G_{18}, G_{19} as self-conjugate subgroups. It is of order 1152 and may be generated by G_{19} and

$$bjofpm.ceg.dnihlk.$$

We have now found the 20 primitive groups of degree 16 that do not contain the alternating group of this degree, and have seen that 10 of them are solvable while the remaining 10 are insolvable. Since the given generating substitutions, excluding H, do not contain a, they generate a subgroup G_1' whose order is obtained by dividing the order of the group by 16. If G_1' is α times transitive, the corresponding group is $\alpha + 1$ times transitive. The class of G_1' is the same as that of the corresponding group, etc. The G_1' of G_{17} is the group of isomorphisms of H. By adding the alternating and the symmetric group to the preceding we obtain the 22 primitive groups of degree 16. The last two are evidently unsolvable. Their factors of composition are respectively $16! \div 2$; 2, $16! \div 2$.

§2.

Proof that there are no other primitive groups of degree sixteen.

As we have examined all the possible groups that contain H as a self-conjugate subgroup and whose orders are divisible by 5, we do not need to consider,

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* Dyck, Mathematische Annalen, vol. 22, p. 102.

in what follows, the groups which satisfy these two conditions. We shall consider the simply transitive and the multiply transitive groups separately, beginning with the former. The group in question will generally be represented by G and its subgroup which contains all its substitutions that do not contain a given element by G'.

A.—*Simply transitive groups.*

We shall make frequent use of the following theorems:

Theorem I. G' cannot contain a transitive subgroup.

Theorem II. All the prime numbers which divide the order of one of the transitive constituents of G', divide the order of every other constituent.

Theorem III. If a transitive constituent of G' is of a prime degree, all its other transitive constituents are of an equal or a larger degree.

Theorem IV. If p^α is the highest power of a prime number that is contained in the order of G, the subgroups of order p^α are transformed by the substitutions of G according to a transitive group whose order is divisible by p^β but not by $p^{\beta+1}$. G contains a self-conjugate subgroup of order $p^{\alpha-\beta}$.

Theorem V. If G' contains a self-conjugate subgroup (H) of degree $n-\alpha$, n being the degree of G, H' must be the transform, with respect to substitutions of G, of $\alpha-1$ other subgroups of G' ($H'_1, H'_2, \ldots, H'_{\alpha-1}$). The substitutions of G that transform H'_β, ($\beta=1, 2, \ldots, \alpha-1$), into H' transform also all the substitutions of G' that are commutative to H'_β into substitutions of G'.

Theorem VI. G' transforms $H'_1, H'_2, \ldots, H'_{\alpha-1}$ according to the elements in one of its constituents of degree $\alpha-1$, and no two of these subgroups can have all their elements in common, nor can any of them contain all the elements of H'.

Theorem VII. The group generated by $H'_1, H'_2, \ldots, H'_{\alpha-1}$ is of degree $n-1$, and G' is a maximal subgroup of G.

Theorem VIII. Every self-conjugate subgroup of a primitive group is transitive.

G' contains a transitive constituent of degree 12.

The other constituent is the symmetric group of degree 3. To 1 in this group must correspond an intransitive subgroup of the constituent of degree 12 (H'). The systems of intransitivity of H' are systems of non-primitivity of the constituent of degree 12. These systems could not be of degree 2 since they would have to be transformed according to a regular group, and H' would then contain all the substitutions of G' whose degree <13.

The given systems could not be of degree 3 since they would have to be transformed according to a transitive group whose order is divisible by 4. As the degree of the systems of H' could evidently not exceed that of a constituent of G', it is impossible to construct a primitive group of degree 16 in which G' contains a transitive constituent of degree 12. From one of the given theorems it follows directly that G' could not contain a transitive constituent of degree 11.

G' contains a transitive constituent of degree 10.

The constituent of degree 5 must clearly be transitive, and its order must be the same as the order of G'; i. e. G' must be obtained by establishing a simple isomorphism between a transitive group of degree 10 and one of degree 5. When the order of G' is 5, 10, or 20, the subgroups of order 2^{α} in G must be transformed by its substitutions according to a transitive group of degree 5. Hence there must be a self-conjugate subgroup of order 2^{β}, $\beta > 0$, in G, according to the given theorem. Since the substitutions of order 5 are of degree 15, we have the congruence

$$2^{\beta} \equiv 1. \qquad \text{mod. } 5.$$

As $\alpha < 8$, $\beta = 4$. We have considered all such groups.

When G' is of order 60 or 120, G must transform its subgroups of order 2^{α} according to some transitive group of degree 5 or 15. In the former case G contains a self-conjugate subgroup of order 16, as we have just proved. In the latter case the corresponding group of degree 15 must be non-primitive, since the orders of the primitive groups are divisible by 9. Since such a non-primitive group could contain no substitution besides identity that leaves all its systems unchanged, it must be simply isomorphic to a transitive group of degree 5. Hence the preceding proof applies also to this case.

G' contains a transitive constituent of degree 9.

Since the order of G' cannot be divisible by 5, the constituent of degree 6 must be either non-primitive or intransitive. If the order of G' would exceed that of the constituent of degree 6, the constituent of degree 9 would be non-primitive and G' would contain an H' of order 3^{α}. As the substitutions of order 3 and degree < 9 in the constituent of degree 9 are commutative, those of order 3 in the constituent of degree 6 could not be commutative, i. e. this constituent would be transitive and it would contain 4 conjugate subgroups of

order 3. As this is clearly impossible, the order of G' is the same as that of the constituent of degree 6.

The order of G' cannot exceed that of the constituent of degree 9, since the order of the quotient group of the constituent of degree 6 with respect to a suitable self-conjugate subgroup would not be divisible by 9. Hence G' must be composed of two simply isomorphic groups whose degrees are 9 and 6. The latter must be transitive, for if it were intransitive all its subgroups of order 3 and degree < 15 would be self-conjugate.

From the preceding it follows that G' must be of order 18, 36, or 72. G must therefore contain 8 conjugate subgroups of order 3 and degree 12. As none of its substitutions besides identity can transform each of these subgroups into itself, it must be simply isomorphic to a transitive group of degree 8 and order 288, 576, or 1152. There are only 4 such transitive groups. We have seen that each of them contains one maximal subgroup that does not include any self-conjugate subgroup besides identity and whose order is obtained by dividing the order of the group by 16. In other words, we have seen that each of these groups is simply isomorphic to one primitive group of degree 16. As each of the given groups of degree 8 contains only one set of conjugate subgroups of the required type, each of them is simply isomorphic to only one primitive group of degree 16.

G' contains a transitive constituent of degree 8.

It is clear that the constituent of degree 7 must be transitive. Since the two transitive constituents of G' would be primitive, they would have to be simply isomorphic. Hence G' would be of order 168. Its substitutions of order 2 would be of degree 12. The 6 systems of such a substitution are transformed according to $\{(abcd)_4(ef)\}$ dim. by the substitutions of G'. The substitutions of G would have to transform these systems according to a transitive group of degree 6 and order 16. This is clearly impossible.

If G' would contain a transitive constituent of degree 7 the other would have to be transitive and of degree 8. We have just seen that this is impossible. We have now considered all the cases when G' contains a constituent whose degree exceeds 6. The remaining cases do not lead to any additional group, and are so simple that it seems unnecessary to consider them here.

B.—*Multiply transitive groups.*

We shall begin with the cases when G' is non-primitive and contains an intransitive self-conjugate subgroup (H_1). If H_1 contains 3 systems of intransitivity it must be composed of three simply isomorphic transitive groups of degree 5, since its order cannot be divisible by 25. If its order is 5, 10, or 20 it must contain a self-conjugate subgroup of order 16, as has been proved above for a similar case. Its order could not be 60 or 120, since the number of its subgroups of order 3 and degree 9 would not be divisible by 7.

If H_1 contains 5 systems of intransitivity its subgroup of order 3^α must satisfy one of the two congruences

$$3^\alpha \equiv 1 \text{ mod. } 5, \qquad 3^\alpha \equiv 3 \text{ mod. } 5,$$

since a substitution of order 5 in H_1 could not transform two of its subgroups of order 3 into themselves. Hence $\alpha = 0$, 1, or 4. If α were 4, H_1 would contain 5 or 10 subgroups of order 3 and degree 6, for only 20 such subgroups are found in the group of order 3^5, and the two of the same degree could not occur in H_1. In G each of these subgroups would have to be transformed into itself by a subgroup whose order is divisible by 5. This is clearly impossible. Hence $\alpha = 0$ or 1.

We have seen that G contains a self-conjugate subgroup of order 16 when $\alpha = 0$ or when $\alpha = 1$, and the systems of intransitivity of H_1 are permuted by G' according to the metacyclic group or one of its subgroups. When H_1 is of order 3 and G' transforms its systems according to the alternating group of degree 5, its 3 substitutions that correspond to a substitution of order 3 in this alternating group must be of order 3, two of them must be of degree 12 and the third of degree 15. All the substitutions of G' must therefore be commutative with each substitution of H_1. Hence all its subgroups of order 12 that do not include any self-conjugate subgroup are conjugate in two sets. Since one of these two simply isomorphic transitive groups* of degree 16 contains substitutions of order 3 and degree 9, we need to consider only one G'. We may suppose that it is generated by

$$adgjmbehkncfilo, \quad adh \,.\, bei \,.\, cfg \,.\, mon.$$

The substitutions of G that transform the latter of these generators into itself must permute its systems according to the alternating group of degree 4,

* Jordan, "Traité des Substitutions," p. 272.

since those of G' permute them according to the alternating group of degree 3. Hence and from the fact that $bc.dh.eg.if.kl.mn$ transforms the given G into itself, we may suppose that G contains one of the following substitutions:

$$\left.\begin{matrix} amdohn \\ andmho \\ aodnhm \end{matrix}\right\} \left\{\begin{matrix} bcefig \\ bgecif \\ bfegic \end{matrix}\right\} \left\{\begin{matrix} jp.kl \\ jk.pl \end{matrix}\right. .$$

By trial we find that only one of these 18 substitutions, viz. $amdohn.bcefig.jp.kl$, generates a group whose substitutions that do not contain p are contained in G'. The cube of this last generating substitution and its transforms form the 15 substitutions differing from identity of a self-conjugate subgroup of order 16. Hence the group which contains the G' that transforms the systems of H_1 according to the symmetric group of degree 5 must also contain a self-conjugate subgroup of order 16.

There could be no primitive group containing the H_1 of order 6 since such a group would have to contain one of the two preceding as self-conjugate subgroup. This is impossible, since the substitutions of degree 12 and order 3 in G' could not transform the negative substitutions of H_1 among themselves. Hence it remains only to consider the cases when G' is a primitive group of degree 15.

There are only 4 such primitive groups that do not include the alternating group, viz. those which are simply isomorphic to the symmetric group of degree 6 and the alternating groups of degrees 6, 7, 8. Each of these groups contains substitutions of degree 12 and order 3. It will not be difficult to find substitutions of G that are not contained in G' and that are commutative to such a substitution. We shall pursue this method to find all the possible groups and then prove that each of them contains a self-conjugate subgroup of order 16.

The G' of order 360 may be generated by

$$abc.efg.ijm.kln, \quad ahcf.bedg.imkn.jl, \quad adjkl.bmnci.eohfg.$$

Since the first of these generators is transformed into itself by 9 substitutions of G' which permute its first three systems cyclically, and all the substitutions of this type are conjugate, G must contain a subgroup of order 36 that transforms it into itself and permutes its systems according to the alternating group of degree 4. Hence it must contain one of the following substitutions:

$$\left.\begin{matrix} aebfcg \\ afbgce \\ agbecf \end{matrix}\right\} \left\{\begin{matrix} ikjlmn \\ iljnmk \\ injkml \end{matrix}\right\} \left\{\begin{matrix} dh.op \\ do.hp \\ dp.ho \end{matrix}\right.$$

As the required substitution must be transformed into its 5^{th} power by $af.be.cg.dh.ij.kl$, it must be one of the three used as factors in the following products:

$$aebfcg\,.\,injkml\,.\,dh\,.\,op \times ahcf\,.\,bedg\,.\,imkn\,.\,jl = adcb\,.\,gh\,.\,jnlm\,.\,op\,,$$
$$afbgce\,.\,injkml\,.\,dh\,.\,op \times ahcf\,.\,bedg\,.\,imkn\,.\,jl = cd\,.\,ehgf\,.\,jnlm\,.\,op\,,$$
$$agbecf\,.\,injkml\,.\,dh\,.\,op \times ahcf\,.\,bedg\,.\,imkn\,.\,jl = abdc\,.\,efhg\,.\,jnlm\,.\,op\,.$$

From these products it follows that only the last can occur in G since G' is of class 12. The transforms of its cube generate a self-conjugate subgroup of order 16.

The G' of order 720 may be generated by the preceding G' and $ae.bf.cg.dh$. Since the squares of the first two of the given three products are not contained in its subgroup of order 48 generated by the first two given generators of the preceding G' and the one just given, there can be only one G that contains this G'. It evidently contains the same self-conjugate subgroup of order 16 as the preceding G. These two groups are doubly transitive. As the remaining two primitive groups of degree 15 that do not contain the alternating group are doubly transitive, the corresponding groups of degree 16 will be triply transitive.

The G' of order 2520 may be generated by

$$aem\,.\,bfn\,.\,gjk\,.\,hil,\quad cek\,.\,dfl\,.\,gin\,.\,hjm\,,\quad akodn\,.\,bfmij\,.\,cgehl.$$

All the substitutions of G' that are similar to the first of these generators are conjugate in G' and each is transformed into itself by 9 substitutions which permute 3 of its systems according to the alternating group. Hence G must contain one of the following substitutions:

$$\left.\begin{matrix} abefmn \\ afenmb \\ anebmf \end{matrix}\right\} \left\{\begin{matrix} ghjikl \\ gijlkh \\ gljhki \end{matrix}\right\} \left\{\begin{matrix} cd\,.\,op \\ co\,.\,dp \\ cp\,.\,do \end{matrix}\right.$$

Since the required substitution must be transformed into its 5^{th} power by the last of the three generators given above, it must be one of the three employed in forming the following products:

$$anebmf\,.\,ghjikl\,.\,cd\,.\,op \times akodn\,.\,bfmij\,.\,cgehl = biopdglefkcnh\,,$$
$$anebmf\,.\,gijlkh\,.\,cd\,.\,op \times akodn\,.\,bfmij\,.\,cgehl = bi\,.\,cnhefklopdgj\,,$$
$$anebmf\,.\,gljkhi\,.\,cd\,.\,op \times akodn\,.\,bfmij\,.\,cgehl = biefkjl\,.\,cnhopdg\,.$$

From these products it follows that only the last can occur in G. The transforms of its cube generate a self-conjugate subgroup of order 16.

The G' of order 20160 may be generated by

$$ab.cd.ef.gh, \quad aem.bfn.gjk.hil, \quad akcgemj.bldhfni, \quad aocgjmk.bdnflih.$$

As all the substitutions of G' that are similar to the second of these generators are conjugate, this G must also contain one of the 27 substitutions given in the preceding case. Since the required substitution must be transformed into its 5th power by $em.fn.gk.hl$, which is contained in G', it must be one of the three used as factor in the following equations:

$$anebmf.gljhki.cd.op \times akcgemj.bldhfni = aiel.bjfk.ch.dg.mn.op,$$
$$anebmf.gljhki.co.dp \times akcgemj.bldhfni = aiel.bjfk.cogdph.mn,$$
$$anebmf.gljhki.cp.do \times akcgemj.bldhfni = aiel.bjfk.cpgdoh.mn.$$

Hence only the first of these three can occur in a G. The transforms of its cube generate a self-conjugate subgroup of order 16. We have now considered all the possible primitive groups of degree 16 that do not contain the alternating group, and have found no group that is not contained in the enumeration of the first section. It may be observed that the substitutions upon which our arguments have been based may be selected in many different ways. As suitable substitutions can readily be found by means of the given generating substitutions, it did not seem necessary to indicate in every case how the particular one that has been employed has been obtained.

It is well known that a solvable primitive group must be of degree p^n, p being a prime number.* The following table gives the enumeration of all these groups whose degree is less than 27:

Degree,	3	4	5	7	8	9	11	13	16	17	19	23
Number of groups,	2	2	3	4	2	7	4	6	10	5	6	4

PARIS, *June*, 1897.

* Cf. Jordan, "Traité des Substitutions," p. 398.